

2-LOCAL DERIVATIONS AND AUTOMORPHISMS ON $B(H)$

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ABSTRACT. The paper is devoted to 2-local derivations and 2-local automorphisms on the algebra $B(H)$ of all bounded linear operators on a Hilbert space H . We prove that every 2-local derivation on $B(H)$ is a derivation. A similar result is obtained for automorphisms.

1. INTRODUCTION

Given an algebra \mathcal{A} , a linear operator $D : \mathcal{A} \rightarrow \mathcal{A}$ is called a *derivation*, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (the Leibniz rule). Each element $a \in \mathcal{A}$ implements a derivation D_a on \mathcal{A} defined as $D_a(x) = [a, x] = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be *inner derivations*. If the element a , implementing the derivation D_a , belongs to a larger algebra \mathcal{B} containing \mathcal{A} , then D_a is called a *spatial derivation* on \mathcal{A} .

There exist various types of linear operators which are close to derivations [6, 7, 12]. In particular R. Kadison [6] has introduced and investigated so-called local derivations on von Neumann algebras and some polynomial algebras.

A linear operator Δ on an algebra \mathcal{A} is called a *local derivation* if given any $x \in \mathcal{A}$ there exists a derivation D (depending on x) such that $\Delta(x) = D(x)$. The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations [6]. In particular Kadison [6] has proved that each continuous local derivation from a von Neumann algebra M into a dual M -bimodule is a derivation.

In 1997, P. Semrl [12] introduced the concepts of 2-local derivations and 2-local automorphisms. A map $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ (not linear in general) is called a *2-local derivation* if for every $x, y \in \mathcal{A}$, there exists a derivation $D_{x,y} : \mathcal{A} \rightarrow \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. A map $\Theta : \mathcal{A} \rightarrow \mathcal{A}$ (not linear in general) is called a *2-local automorphism* if for every $x, y \in \mathcal{A}$, there exists an automorphism $\Phi_{x,y} : \mathcal{A} \rightarrow \mathcal{A}$ such that $\Theta(x) = \Phi_{x,y}(x)$ and $\Theta(y) = \Phi_{x,y}(y)$. Local and 2-local maps have been studied on different operator algebras by many authors [1, 2, 4, 6–13].

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In [12], P. Semrl described 2-local derivations and automorphisms on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H . A similar description for the finite-dimensional case appeared later in [7], [11]. In the paper [9] 2-local derivations and automorphisms have been described on matrix algebras over finite-dimensional division rings.

In the present paper we suggest a new technique and generalize the above mentioned results of [12] and [7] for arbitrary Hilbert spaces. Namely we consider 2-local derivations and 2-local automorphisms on the algebra $B(H)$ of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space H . We prove that every 2-local derivation on $B(H)$ is a derivation. A similar result is obtained for automorphisms, strengthening a result of [10] in the case of Hilbert spaces.

2. MAIN RESULTS

Let H be an arbitrary Hilbert space, and let $B(H)$ be the algebra of all linear bounded operators on H . Denote by $\mathcal{F}(H)$ the ideal of all finite-dimensional operators from $B(H)$ and by tr the canonical trace on $B(H)$.

Note that the algebra $\mathcal{F}(H)$ is semi-prime, i.e. if $a \in \mathcal{F}(H)$ and $a\mathcal{F}(H)a = \{0\}$ then $a = 0$. Indeed, let $a \in \mathcal{F}(H)$ and $a\mathcal{F}(H)a = \{0\}$, i.e. $axa = 0$ for all $x \in \mathcal{F}(H)$. In particular for $x = a^*$ we have $aa^*a = 0$ and hence $a^*aa^*a = 0$, i.e. $|a|^4 = 0$. Therefore $a = 0$.

Further any derivation D on $B(H)$ maps the ideal $\mathcal{F}(H)$ into itself. Indeed, for any $x \in \mathcal{F}(H)$ there exists a projection $p \in \mathcal{F}(H)$ such that $x = xp$. Then

$$D(x) = D(xp) = D(x)p + xD(p),$$

and therefore $D(x) \in \mathcal{F}(H)$. Hence any 2-local derivation on $B(H)$ also maps $\mathcal{F}(H)$ into itself. Similarly every automorphism on $B(H)$ also maps $\mathcal{F}(H)$ into itself.

Lemma 2.1. *Let $b \in B(H)$ be an arbitrary element. If $\text{tr}(xb) = 0$ for all $x \in \mathcal{F}(H)$ then $b = 0$.*

Proof. Let $b \in B(H)$. For any finite-dimensional projection $e \in B(H)$ we have $eb^* \in \mathcal{F}(H)$ and therefore by the assumption of the lemma it follows that $\text{tr}(eb^*b) = 0$. Thus

$$0 = \text{tr}(eb^*b) = \text{tr}(e^2b^*b) = \text{tr}(eb^*be) = \text{tr}((be)^*(be)),$$

i.e.

$$\text{tr}((be)^*(be)) = 0.$$

Since the trace tr is faithful, we obtain $(be)^*(be) = 0$, i.e. $be = 0$.

Now take a family of mutually orthogonal one-dimensional projections $\{e_\alpha\}_{\alpha \in J}$ in $B(H)$ such that $\bigvee_{\alpha \in J} e_\alpha = \mathbf{1}$. Given a finite subset $F \subset J$ put $e_F = \sum_{\alpha \in F} e_\alpha$. We obtain an increasing net $\{e_F\}$, when F runs over all finite subsets of J . Since $e_F \uparrow \mathbf{1}$ we have that

$$0 = be_Fb^* \uparrow bb^*,$$

i.e. $bb^* = 0$. Thus $b = 0$. The proof is complete. \square

Lemma 2.2. *If $\Delta : B(H) \rightarrow B(H)$ is a 2-local derivation such that $\Delta|_{\mathcal{F}(H)} \equiv 0$, then $\Delta \equiv 0$.*

Proof. Let $\Delta : B(H) \rightarrow B(H)$ be a 2-local derivation such that $\Delta|_{\mathcal{F}(H)} \equiv 0$. For arbitrary $x \in B(H)$ and $y \in \mathcal{F}(H)$ there exists a derivation $D_{x,y}$ on $B(H)$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. By [5, Corollary 3.4] there exists element $a \in B(H)$ such that

$$[a, xy] = D_{x,y}(xy) = D_{x,y}(x)y + xD_{x,y}(y) = \Delta(x)y + x\Delta(y),$$

i.e.

$$[a, xy] = \Delta(x)y + x\Delta(y).$$

Since $y \in \mathcal{F}(H)$ we have $\Delta(y) = 0$, and therefore $[a, xy] = \Delta(x)y$. Since the trace tr accepts finite values on $\mathcal{F}(H)$ and $\mathcal{F}(H)$ is an ideal in $B(H)$ we have

$$\text{tr}(axy) = \text{tr}((ax)y) = \text{tr}(y(ax)) = \text{tr}((ya)x) = \text{tr}(x(ya)) = \text{tr}(xya).$$

Thus

$$0 = \text{tr}(axy - xya) = \text{tr}([a, xy]) = \text{tr}(\Delta(x)y),$$

i.e. $\text{tr}(\Delta(x)y) = 0$ for all $y \in \mathcal{F}(H)$. By Lemma 2.1 we have that $\Delta(x) = 0$. The proof is complete. \square

The following theorem is the main result of this paper.

Theorem 2.3. *Let H be an arbitrary Hilbert space, and let $B(H)$ be the algebra of all bounded linear operators on H . Then every 2-local derivation $\Delta : B(H) \rightarrow B(H)$ is a derivation.*

Proof. Let $\Delta : B(H) \rightarrow B(H)$ be a 2-local derivation. For each $x, y \in \mathcal{F}(H)$ there exists a derivation $D_{x,y}$ on $B(H)$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. By [5, Corollary 3.4] there exists an element $a \in B(H)$ such that

$$[a, xy] = D_{x,y}(xy) = D_{x,y}(x)y + xD_{x,y}(y) = \Delta(x)y + x\Delta(y),$$

i.e.

$$[a, xy] = \Delta(x)y + x\Delta(y).$$

Similarly as in Lemma 2.2 we have

$$0 = \text{tr}(axy - xya) = \text{tr}([a, xy]) = \text{tr}(\Delta(x)y + x\Delta(y)),$$

i.e. $\text{tr}(\Delta(x)y) = -\text{tr}(x\Delta(y))$. For arbitrary $u, v, w \in \mathcal{F}(H)$, set $x = u + v$, $y = w$. Then from above we obtain

$$\begin{aligned} \text{tr}(\Delta(u + v)w) &= -\text{tr}((u + v)\Delta(w)) = \\ &= -\text{tr}(u\Delta(w)) - \text{tr}(v\Delta(w)) = \text{tr}(\Delta(u)w) + \text{tr}(\Delta(v)w) = \text{tr}((\Delta(u) + \Delta(v))w), \end{aligned}$$

and so

$$\text{tr}((\Delta(u + v) - \Delta(u) - \Delta(v))w) = 0$$

for all $u, v, w \in \mathcal{F}(H)$. Denote $b = \Delta(u + v) - \Delta(u) - \Delta(v)$ and put $w = b^*$. Then $\text{tr}(bb^*) = 0$. Since the trace tr is faithful it follows that $bb^* = 0$, i.e. $b = 0$. Therefore

$$\Delta(u + v) = \Delta(u) + \Delta(v),$$

i.e. Δ is an additive map on $\mathcal{F}(H)$.

Now let us show that Δ is homogeneous. Indeed, for each $x \in B(H)$, and for $\lambda \in \mathbf{C}$ there exists a derivation $D_{x,\lambda x}$ such that $\Delta(x) = D_{x,\lambda x}(x)$ and $\Delta(\lambda x) = D_{x,\lambda x}(\lambda x)$. Then

$$\Delta(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \Delta(x).$$

Hence, Δ is homogenous and therefore it is a linear operator.

Finally, for each $x \in B(H)$, there exists a derivation D_{x,x^2} such that $\Delta(x) = D_{x,x^2}(x)$ and $\Delta(x^2) = D_{x,x^2}(x^2)$. Then

$$\Delta(x^2) = D_{x,x^2}(x^2) = D_{x,x^2}(x)x + xD_{x,x^2}(x) = \Delta(x)x + x\Delta(x)$$

for all $x \in B(H)$. Therefore, the restriction $\Delta|_{\mathcal{F}(H)}$ of the operator Δ on $\mathcal{F}(H)$ is a linear Jordan derivation on $\mathcal{F}(H)$ in the sense of [3]. In [3, Theorem 1] it is proved that any Jordan derivation on a semi-prime algebra is a derivation. Since $\mathcal{F}(H)$ is semiprime, therefore the linear operator $\Delta|_{\mathcal{F}(H)}$ is a derivation on $\mathcal{F}(H)$.

Now by [5, Theorem 3.3] the derivation $\Delta|_{\mathcal{F}(H)} : \mathcal{F}(H) \rightarrow \mathcal{F}(H)$ is spatial, i.e.

$$\Delta(x) = ax - xa, x \in \mathcal{F}(H) \quad (2.1)$$

for an appropriate $a \in B(H)$.

Let us show that $\Delta(x) = ax - xa$ for all $x \in B(H)$. Consider the 2-local derivation $\Delta_0 = \Delta - D_a$. Then from the equality (2.1) we obtain that $\Delta_0|_{\mathcal{F}(H)} \equiv 0$. Now by Lemma 2.2 it follows that $\Delta_0 \equiv 0$. This means that $\Delta = D_a$. The proof is complete. \square

For 2-local automorphisms of $B(H)$ we have a similar result.

Theorem 2.4. *Every 2-local automorphism $\Theta : B(H) \rightarrow B(H)$ is an automorphism.*

Proof. Since every automorphism on $B(H)$ maps $\mathcal{F}(H)$ into itself it follows that the restriction $\Theta|_{\mathcal{F}(H)}$ is a 2-local automorphism on $\mathcal{F}(H)$. By [10, Theorem 2.5] $\Theta|_{\mathcal{F}(H)}$ is an automorphism. Therefore by [5, Theorem 3.1] there exists an invertible element $a \in B(H)$ such that $\Theta(x) = axa^{-1}$ for all $x \in \mathcal{F}(H)$.

Let us show that in fact $\Theta(x) = axa^{-1}$ for all $x \in B(H)$. Consider the 2-local automorphism $\Phi(x) = a^{-1}\Theta(x)a$, $x \in B(H)$. It is clear that $\Phi(x) = x$ for all $x \in \mathcal{F}(H)$.

Now for each $x \in B(H)$ and $y \in \mathcal{F}(H)$ there exists an automorphism $\Psi_{x,y}$ such that $\Phi(x) = \Psi_{x,y}(x)$ and $\Phi(y) = \Psi_{x,y}(y)$. By [5, Corolarry 3.2] there exists an invertible element $b \in B(H)$ such that

$$b(xy)b^{-1} = \Psi_{x,y}(xy) = \Psi_{x,y}(x)\Psi_{x,y}(y) = \Phi(x)y.$$

Thus

$$\text{tr}(xy) = \text{tr}(b(xy)b^{-1}) = \text{tr}(\Phi(x)y),$$

i.e. $\text{tr}([\Phi(x) - x]y) = 0$ for all $y \in \mathcal{F}(H)$. By Lemma 2.1 we have that $\Phi(x) = x$. This means that $\Theta(x) = axa^{-1}$ for all $x \in B(H)$. The proof is complete. \square

Remark 2.5. A similar result for 2-local automorphisms on $B(X)$, where X is a locally convex space, has been obtained in [10, Corolarry 2.6] under the additional assumption of continuity of the map with respect to the weak operator topology.

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